Convergence Analysis of a Stereophonic Acoustic Echo Canceller Part I: Convergence Characteristics of Tap Weights

ステレオエコーキャンセラの収束解析 I タップ重みの収束特性

Akihiro Hirano *and* Shin'ichi Koike†

平野 晃宏　小池 伸一

Information Technology Research Laboratories

Engineering Coordination and Planning Division†,

NEC Corporation

日本電気株式会社

情報メディア研究所　技術企画部†

ABSTRACT

This paper analyzes convergence characteristics of a stereophonic acoustic echo canceller for strongly cross-correlated input signals. One of the two reference input signals to the adaptive filters is assumed to be a delayed and attenuated version of the other signal. Convergence of tap weights and the convergence condition for mean tap-weights are analyzed. Analytical results show that a part of the tap weights does not converge to their optimum value, i.e., the impulse response of the echo paths. Computer simulation results confirm the analyses.

あらまし

線形結合型ステレオエコーキャンセラを対象とした、収束特性の解析結果を報告する。遠端側で同時に発声する話者が1人である場合を想定して、片チャンネルの参照入力信号が、もう一方の参照入力信号に遅延と振幅差を与えたものとして表されるモデルを用いる。タップ重みの期待値を解析し、その収束過程および最終収束値を導出する。タップ重みは、最適値であるエコーパスのインパルス応答には収束しない。計算機シミュレーションにより、解析の妥当性を示す。

1. Introduction

Echo cancellers are used to reduce echoes in a wide range of applications, such as TV conference systems and hands-free telephones. To realistic TV conference systems, multi-channel audio, at least stereophonic, is essential. For stereophonic teleconference systems, stereophonic acoustic echo cancellers have been studied[1-6].

In stereophonic echo cancellers, the influence of strong cross-correlation between two input signals is largest problem[6, 7]. However, convergence of stereophonic echo cancellers for cross-correlated input signals has not been studied in details. Reported convergence analysis is only for a very simple case in which one of the two reference input signals to the adaptive filters is a delayed version of the other signal[6]. The mean squared error (MSE) and the convergence condition of the MSE have not been analyzed.

This paper investigates convergence characteristics of a stereophonic acoustic echo canceller for strongly cross-correlated input signals. One of the two reference inputs is assumed to be a delayed and attenuated version of the other. Convergence of tap weights, the convergence condition for mean tap-weights are analyzed. Computer simulation results will validate the analyses. Part II

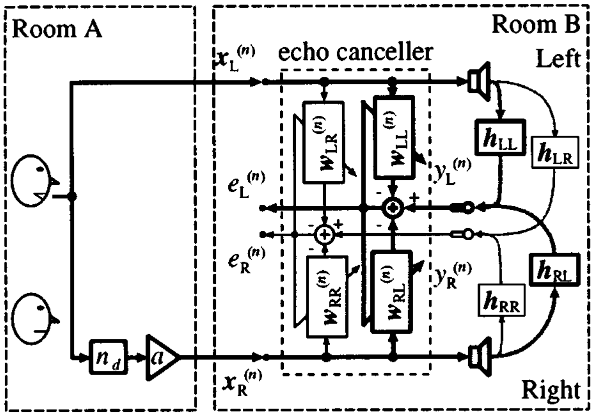


Fig. 1. Stereophonic teleconferencing using echo canceller.

will analyze the MSE, the convergence condition of the MSE, and the error surface[8]. The effects of dither insertion will also be shown.

2. Stereophonic Acoustic Echo Canceller

Let us concentrate on a stereophonic acoustic echo canceller based on linear combination[1]. Figure 1 depicts a schematic block diagram of a stereophonic teleconferencing using this echo canceller. The echo canceller in Room B consists of four adaptive filters corresponding to four echo paths from two loudspeakers to two microphones. Each adaptive filter estimates the impulse response of the corresponding echo path.

The input signal vector $\boldsymbol{x}^{(n)}$ is defined by

$$\boldsymbol{x}^{(n)} = \begin{bmatrix} \boldsymbol{x}_L^{(n)} \\ \boldsymbol{x}_R^{(n)} \end{bmatrix} \tag{1}$$

where n is the time index. $\boldsymbol{x}^{(n)}$ consists of the input signal vectors for the left channel, $\boldsymbol{x}_L^{(n)}$, and that for the right, $\boldsymbol{x}_R^{(n)}$. The subscripts L and R denote "Left" and "Right" channels, respectively. Tap-weight matrix $\mathbf{W}^{(n)}$ is given by

$$\mathbf{W}^{(n)} = \begin{bmatrix} \boldsymbol{w}_{LL}^{(n)} & \boldsymbol{w}_{LR}^{(n)} \\ \boldsymbol{w}_{RL}^{(n)} & \boldsymbol{w}_{RR}^{(n)} \end{bmatrix} \tag{2}$$

where $\boldsymbol{w}_{LL}^{(n)}$, $\boldsymbol{w}_{RL}^{(n)}$, $\boldsymbol{w}_{LR}^{(n)}$ and $\boldsymbol{w}_{RR}^{(n)}$ are tap-weight vectors for four adaptive filters. Using the echo-path impulse-response matrix $\mathbf{H}$ defined by

$$\mathbf{H} = \begin{bmatrix} \boldsymbol{h}_{LL} & \boldsymbol{h}_{LR} \\ \boldsymbol{h}_{RL} & \boldsymbol{h}_{RR} \end{bmatrix}, \tag{3}$$

the tap-weight error matrix $\Theta^{(n)}$ is given by

$$\Theta^{(n)} = \mathbf{H} - \mathbf{W}^{(n)}. \tag{4}$$

The error vector $\boldsymbol{e}^{(n)}$ is calculated by

$$\boldsymbol{e}^{(n)} = \begin{bmatrix} e_L(n) \\ e_R(n) \end{bmatrix}$$

$$= \Theta^{(n)T} \boldsymbol{x}^{(n)} \tag{5}$$

where $[\cdot]^T$ denotes transpose of a matrix $[\cdot]$.

Assuming the LMS (Least Mean Squares) algorithm[9], the tap-weight matrix $\mathbf{W}^{(n)}$ is updated by

$$\mathbf{W}^{(n+1)} = \mathbf{W}^{(n)} + \mu \boldsymbol{x}^{(n)}(\boldsymbol{e}^{(n)T} + \boldsymbol{v}^{(n)T}). \tag{6}$$

In (6), a positive constant μ is a step-size which controls the convergence. $\boldsymbol{v}^{(n)}$ is an additive noise vector defined by

$$\boldsymbol{v}^{(n)} = \begin{bmatrix} v_L(n) \\ v_R(n) \end{bmatrix}. \tag{7}$$

The additive noise $\boldsymbol{v}^{(n)}$ is assumed to be independent of the input signal $\boldsymbol{x}^{(n)}$.

3. Convergence Analysis of Averaged Tap-Weight Error

In the following analysis, a single-talker case is assumed in which one of the attendants in Room A is speaking. Both signals sent to Room B contain the same speech signal. Thus cross-correlated signals are generated. Note that such a situation is commonly encountered in many teleconferences.

The relation between two input signals is given by

$$\boldsymbol{x}_R^{(n)} = a \boldsymbol{x}_L^{(n-n_d)} \tag{8}$$

where n_d is a time delay between $x_L^{(n)}$ and $\boldsymbol{x}_R^{(n)}$, a is an attenuation factor. $\boldsymbol{x}^{(n)}$ is also assumed to be zero mean, white Gaussian process with a variance of σ_X^2 and to be independent of $\Theta^{(n)}$. The independent assumption of $x^{(n)}$ and $\Theta^{(n)}$ is common in many analyses and is valid if the step size μ is small[9].

The ensemble average of the tap-weight error matrix defined by

$$\mathbf{M}^{(n)} = E[\Theta^{(n)}] \tag{9}$$

will be analyzed. Updating equation for $\Theta^{(n)}$ is derived from (4), (5), and (6) as

$$\Theta^{(n+1)} = \Theta^{(n)} - \mu x^{(n)} x^{(n)T} \Theta^{(n)} - \mu x^{(n)} v^{(n)T}. \tag{10}$$

By taking an ensemble average of (10), difference equation for $\mathbf{M}^{(n)}$ becomes

$$\mathbf{M}^{(n+1)} = (\mathbf{I}_{2N} - \mu\mathbf{R})\mathbf{M}^{(n)} \tag{11}$$

where $\mathbf{I}_{2N}$ is a $2N \times 2N$ unit matrix, $\mathbf{R}$ is an "extended" covariance matrix defined by

$$\mathbf{R} = E[x^{(n)} x^{(n)T}]. \tag{12}$$

$\mathbf{R}$ consists of both the auto-covariance matrices, $E[x_L^{(n)} x_L^{(n)T}]$ and $E[x_R^{(n)} x_R^{(n)T}]$, and the cross-covariance $E[x_L^{(n)} x_R^{(n)T}]$.

For input signals shown in (8), $\mathbf{R}$ becomes

$$\mathbf{R} = \begin{bmatrix} \sigma_X{}^2\mathbf{I}_{n_d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sigma_X{}^2\mathbf{I}_{N-n_d} & a\sigma_X{}^2\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & a\sigma_X{}^2\mathbf{I}_{N-n_d} & a^2\sigma_X{}^2\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a^2\sigma_X{}^2\mathbf{I}_{n_d} \end{bmatrix} \tag{13}$$

where $\mathbf{0}$ is a zero matrix. Let us concentrate on a $2(N - n_d) \times 2(N - n_d)$ matrix located in the center of $\mathbf{R}$, i.e.,

$$\mathbf{R}' = \begin{bmatrix} \sigma_X{}^2\mathbf{I}_{N-n_d} & a\sigma_X{}^2\mathbf{I}_{N-n_d} \\ a\sigma_X{}^2\mathbf{I}_{N-n_d} & a^2\sigma_X{}^2\mathbf{I}_{N-n_d} \end{bmatrix}. \tag{14}$$

It is clear that $\mathbf{R}'$ is singular because multiplying the upper row of (14) by a generates the lower row. Two rows are linearly dependent.

By introducing an orthonormal matrix

$$\mathbf{P} = \begin{bmatrix} \mathbf{I}_{n_d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{\sqrt{1+a^2}}\mathbf{I}_{N-n_d} & \frac{a}{\sqrt{1+a^2}}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \frac{-a}{\sqrt{1+a^2}}\mathbf{I}_{N-n_d} & \frac{1}{\sqrt{1+a^2}}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_d} \end{bmatrix}, \tag{15}$$

$\mathbf{R}$ can be diagonalized as

$$\Lambda = \mathbf{PRP}^{-1}$$

$$= \begin{bmatrix} \sigma_X{}^2\mathbf{I}_{n_d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & (1+a^2)\sigma_X{}^2\mathbf{I}_{N-n_d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & a^2\sigma_X{}^2\mathbf{I}_{n_d} \end{bmatrix}. \tag{16}$$

Derivation of P is given in Appendix A. Obviously, the eigenvalues of $\mathbf{R}$ are $\sigma_X{}^2$, $(1+a^2)\sigma_X{}^2$, $a^2\sigma_X{}^2$, and 0. Since $\mathbf{R}$ is singular, the averaged tap-weight error $\mathbf{M}$ can not converge to $\mathbf{0}$ even if the convergence condition is satisfied.

Introduction of a new variable

$$\tilde{\mathbf{M}}^{(n)} = \mathbf{PM}^{(n)} \tag{17}$$

leads to a simplified difference equation as

$$\tilde{\mathbf{M}}^{(n+1)} = (\mathbf{I}_{2N} - \mu\Lambda)\tilde{\mathbf{M}}^{(n)}. \tag{18}$$

Since $(\mathbf{I}_{2N} - \mu\Lambda)$ is diagonal, $\tilde{\mathbf{M}}^{(n)}$ can easily be solved as

$$\tilde{\mathbf{M}}^{(n)} = (\mathbf{I}_{2N} - \mu\Lambda)^n\tilde{\mathbf{M}}^{(0)}$$

$$= \begin{bmatrix} \alpha_1{}^n\mathbf{I}_{n_d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha_2{}^n\mathbf{I}_{N-n_d} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha_3{}^n\mathbf{I}_{n_d} \end{bmatrix}\tilde{\mathbf{M}}^{(0)} \tag{19}$$

where α_1, α_2 and α_3 are defined by

$$\alpha_1 = 1 - \mu\sigma_X{}^2 \tag{20}$$

$$\alpha_2 = 1 - \mu(1+a^2)\sigma_X{}^2 \tag{21}$$

$$\alpha_3 = 1 - \mu a^2\sigma_X{}^2. \tag{22}$$

The solution for $\mathbf{M}^{(n)}$ is derived from (17) and (19) as

$$\mathbf{M}^{(n)} = \mathbf{P}^{-1}\tilde{\mathbf{M}}^{(n)}$$

$$= \mathbf{P}^{-1}(\mathbf{I}_{2N} - \mu\Lambda)^n\mathbf{PM}^{(0)}$$

$$= \begin{bmatrix} \alpha_1{}^n \mathbf{I}_{n_d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{\alpha_2{}^n + a^2}{1+a^2}\mathbf{I}_{N-n_d} & a\frac{\alpha_2{}^n - 1}{1+a^2}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & a\frac{\alpha_2{}^n - 1}{1+a^2}\mathbf{I}_{N-n_d} & \frac{\alpha_2{}^n a^2 + 1}{1+a^2}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \alpha_3{}^n \mathbf{I}_{n_d} \end{bmatrix}$$

$$\times \mathbf{M}^{(0)}. \tag{23}$$

Appendix B describes detailed derivation of (23). Finally, the ensemble average of the tap-weight matrix is calculated by

$$E[\mathbf{W}^{(n)}] = \mathbf{H} - \mathbf{M}^{(n)}. \tag{24}$$

The convergence condition for the averaged tap-weight error $\mathbf{M}^{(n)}$ is derived from (20), (21), (22) and (23). All of $\alpha_1{}^\infty$, $\alpha_2{}^\infty$, $\alpha_3{}^\infty$ should be 0. The convergence condition is

$$0 < \mu < \frac{2}{(1+a^2)\sigma_X{}^2}. \tag{25}$$

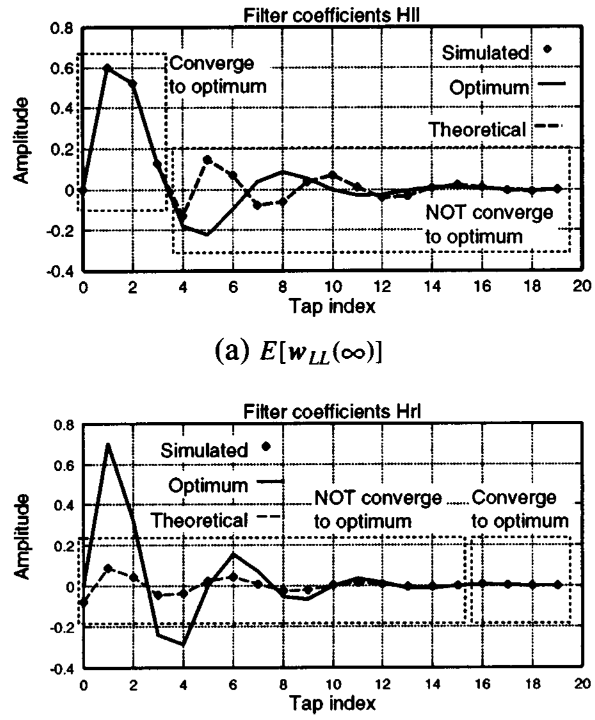
If μ satisfies this condition, $\mathbf{M}^{(n)}$ converges to

$$\mathbf{M}^{(\infty)} = \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{a^2}{1+a^2}\mathbf{I}_{N-n_d} & \frac{-a}{1+a^2}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \frac{-a}{1+a^2}\mathbf{I}_{N-n_d} & \frac{1}{1+a^2}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{M}^{(0)} \tag{26}$$

Obviously, the averaged tap-weight error $\mathbf{M}^{(n)}$ do not converge to $\mathbf{0}$, i.e., the tap-weight matrix $\mathbf{W}^{(n)}$ do not converge to its optimum value $\mathbf{H}$. The convergence values depend on initial value $\mathbf{W}^{(0)}$, n_d and a. If the initial value of $\mathbf{W}^{(n)}$ is $\mathbf{0}$, the final tap weights $\mathbf{W}^{(\infty)}$ becomes

$$E[\mathbf{W}^{(\infty)}] = \begin{bmatrix} \mathbf{I}_{n_d} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \frac{1}{1+a^2}\mathbf{I}_{N-n_d} & \frac{a}{1+a^2}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \frac{a}{1+a^2}\mathbf{I}_{N-n_d} & \frac{a^2}{1+a^2}\mathbf{I}_{N-n_d} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I}_{n_d} \end{bmatrix} \mathbf{H}. \tag{27}$$

(27) shows that some elements of $\mathbf{W}^{(n)}$, upper n_d elements of $w_{LL}{}^{(n)}$ and $w_{LR}{}^{(n)}$, and lower n_d elements of $w_{RL}{}^{(n)}$ and $w_{RR}{}^{(n)}$, converge to the optimum. The other elements



(a) $E[w_{LL}(\infty)]$

(b) $E[w_{RL}(\infty)]$

Fig. 2. Tap weights after converge(1).

converge to non-optimum values.

4. Computer Simulations

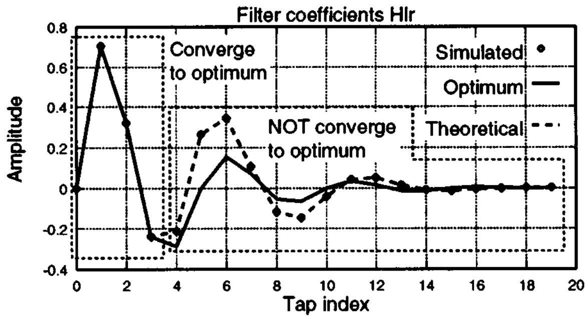
Computer simulations have been carried out and their results have been compared with the analytical results. The number of taps N was 20. The echo paths were given by

$$h_{LL,i} = h_{RR,i} = e^{-0.3i}\sin(0.3\pi i) \tag{28}$$

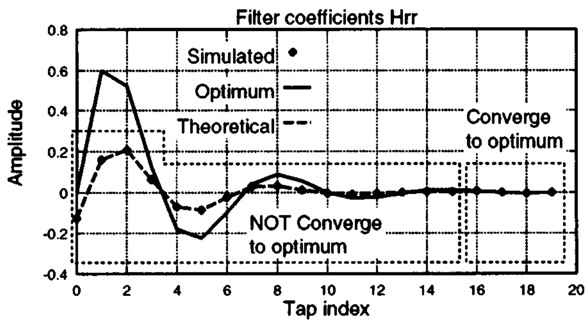
$$h_{RL,i} = h_{LR,i} = e^{-0.3i}\sin(0.4\pi i) \tag{29}$$

where $h_{LL,i}$, $h_{RR,i}$, $h_{RL,i}$ and $h_{LR,i}$ are the i-th element of the echo path vectors $\boldsymbol{h}_{LL}$, $\boldsymbol{h}_{RR}$, $\boldsymbol{h}_{RL}$ and $\boldsymbol{h}_{LR}$, respectively.

The left-channel input signal was white-Gaussian process with unit variance. Parameters a and n_d were selected as 0.6 and 4, respectively. Independent white-Gaussian noises have been added to the echoes as the additive noise. The variance of the additive noise was 0.01, thus the echo-to-noise ratio is about 25dB. The step size μ was settled as 0.01, which satisfies the convergence condition for the MSE shown in part II [8]. An average of 1000 independent runs has been



(a) $E[\mathbf{w}_{LR}(\infty)]$



(b) $E[\mathbf{w}_{RR}(\infty)]$

Fig. 2. Tap weights after converge(2).

calculated.

Averaged tap-weight vectors after convergence are shown in Fig. 2. As shown by the analysis, $4n_d$ tap weights converge to the optimum values. However, all the other weights do not converge to the optimum. The convergence values agree with analytical results shown in (27).

Figure 3 depicts convergence characteristics of two averaged tap-weights. Trajectories for $E[w_{LL,1}(n)]$, the second element of $E[\mathbf{w}_{LL}{}^{(n)}]$, and $E[w_{LL,5}]$, the sixth element, are shown. $E[w_{LL,1}(n)]$ converges to the optimum value while $E[w_{LL,5}]$ does not. The trajectories of averaged tap-weights agree with that derived from (24).

5. Conclusion

A convergence analysis for a stereophonic acoustic echo canceller has been presented. Convergence of averaged tap-weights and the convergence condition for averaged tap-weights have been analyzed for a single talker case, in which one of the two reference input signals to the adaptive filters

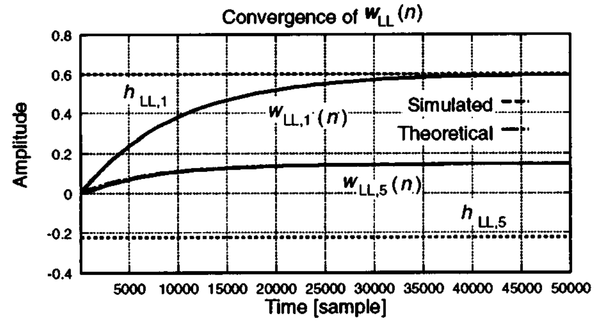


Fig. 3. Convergence of tap weights.

is assumed to be a delayed and attenuated version of the other. Analytical results show that a part of the tap weights does not converge to their optimum value, i.e., the impulse response of the echo paths. Computer simulation results confirm the analysis. Analyses of the MSE, the convergence condition of the MSE, and the error surface will be shown in part II. The effects of dither insertion will also be shown.

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Appendix A. Derivation of P

A part of $\mathbf{R}$ which requires diagonalization, i.e.,

$$\mathbf{R}' = \begin{bmatrix} \sigma_X{}^2\mathbf{I}_{N-n_d} & a\sigma_X{}^2\mathbf{I}_{N-n_d} \\ a\sigma_X{}^2\mathbf{I}_{N-n_d} & a^2\sigma_X{}^2\mathbf{I}_{N-n_d} \end{bmatrix} \qquad \text{(A1)}$$

is considered. Diagonalization of $\mathbf{R}'$ can further be reduced to diagonalization of 2×2 matrix defined by

$$R'' = \sigma_X{}^2\begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix}. \qquad \text{(A2)}$$

The characteristic equation for R'' is given by

$$\det\left(\sigma_X{}^2\begin{bmatrix} 1 & a \\ a & a^2 \end{bmatrix} - \lambda\mathbf{I}\right) = 0. \qquad \text{(A3)}$$

The eigenvalues are 0, $\sigma_X{}^2(1+a^2)$. The normalized eigenvectors are derived as

$$\frac{1}{\sqrt{1+a^2}}\begin{bmatrix} 1 \\ a \end{bmatrix} \qquad \text{(A4)}$$

and

$$\frac{1}{\sqrt{1+a^2}}\begin{bmatrix} -a \\ 1 \end{bmatrix}. \qquad \text{(A5)}$$

Therefore, orthonormal matrix which diagonalize (A2) is given by

$$\frac{1}{\sqrt{1+a^2}}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix}. \qquad \text{(A6)}$$

Orthonormal matrix P shown in (15) is derived by expanding (A6).

Appendix B. Derivation of $\mathbf{M}^{(n)}$

From (17) and (19), the tap-weight error matrix $\mathbf{M}^{(n)}$ is calculated by

$$\begin{aligned} \mathbf{M}^{(n)} &= \mathbf{P}^{-1}\tilde{\mathbf{M}}^{(n)} \\ &= \mathbf{P}^{-1}(\mathbf{I}_{2N} - \mu\Lambda)^n\tilde{\mathbf{M}}^{(0)} \\ &= \mathbf{P}^{-1}(\mathbf{I}_{2N} - \mu\Lambda)^n\mathbf{P}\mathbf{P}^{-1}\tilde{\mathbf{M}}^{(0)} \\ &= \mathbf{P}^{-1}(\mathbf{I}_N - \mu\Lambda)^n\mathbf{P}\mathbf{M}^{(0)} \end{aligned} \qquad \text{(B1)}$$

Calculation of a $2N\times 2N$ matrix $\mathbf{P}^{-1}(\mathbf{I}_N - \mu\Lambda)^n\mathbf{P}$ can be simplified to calculation of 2×2 matrix as

$$\begin{aligned} &\frac{1}{1+a^2}\begin{bmatrix} 1 & -a \\ a & 1 \end{bmatrix}\begin{bmatrix} \alpha_2{}^n & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \\ &= \frac{1}{1+a^2}\begin{bmatrix} \alpha_2{}^n & -a \\ a\alpha_2{}^n & 1 \end{bmatrix}\begin{bmatrix} 1 & a \\ -a & 1 \end{bmatrix} \\ &= \frac{1}{1+a^2}\begin{bmatrix} \alpha_2{}^n + a^2 & a(\alpha_2{}^n - 1) \\ a(\alpha_2{}^n - 1) & a^2\alpha_2{}^n + 1 \end{bmatrix}. \end{aligned} \qquad \text{(B2)}$$

Expanding (B2) leads to (23).