Stabilizing Fast Newton Filters by Using Order-Update Fast Least Squares Algorithm

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Abstract—It is known that the fast Newton transversal filter (FNTF) algorithms suffer from the numerical instability problem if the predictor used for extending the gain vector is calculated by using the fast RLS (FRLS) algorithms. In order to overcome this difficulty, we propose in this paper to combine the FNTF with the order-update FRLS algorithm (we call it predictor based least squares (PLS) algorithm). Very few investigations are reported concerning the numerical property of the PLS algorithm. In the paper, we prove that three main instability sources encountered in both the RLS and FRLS algorithms, including the unstable behavior of the conversion factor, the loss of symmetry and the loss of positive definiteness of the inverse correlation matrix, do not exist in both the PLS algorithm and the combination of the PLS and the FNTF algorithm. As a result, the combination of the PLS and the FNTF algorithm can provide a much more stable and robust numerical performance compared with other combinations, for example the FRLS or the RLS with the FNTF algorithms.

1 Introduction

The FNTF algorithms attract many attentions these years. The main advantage of the FNTF algorithms is the fast computation of the gain vector needed for the adaptation of the transversal filters. Assuming the input signal to be autoregressive of order M, where M is possible to be selected much smaller than the order N of the adaptive filter, then the gain vector can be extended from M to N based on the predictor and the gain vector of order M without sacrificing the performance. The computational savings by using the FNTF algorithms can be significant in some applications like acoustic echo canceler, in which N is usually much greater than M [1].

Like any other fast version of the RLS algorithms, the FNTF algorithms also suffer from the numerical instability problem, if the predictor used for extending the gain vector is calculated by using the FRLS algorithms. The instability of the FRLS algorithms is mainly produced by a hyperbolic rotation (causing the eigenvalues to go out of the unit circle) that has to be operated on the backward predictor in order to obtain the recursive equations for computing the gain vector [2],[3].

In the FRLS algorithms, however, if we assume that the recursions involve both order- and time-update, then the least squares solution can be obtained by using either forward or backward predictor. Therefore, the stable structures of both forward and backward predictors are remained. This leads to the algorithms we called the PLS that include the forward PLS (FPLS) and the backward PLS (BPLS) algorithms. Very few investigations are reported in the literature concerning the numerical property of the PLS algorithms. In [4], we gave a comparative study on the numerical performances of the BPLS and the RLS algorithms and showed that the BPLS algorithm can provide a much more stable and robust numerical performance than that of the RLS algorithm.

In this paper, we present a numerical study on the combination of the BPLS and the FNTF algorithms. Specifically, three main instability sources reported in both the RLS and the FRLS algorithms, including the unstable behavior of the conversion factor, the loss of symmetry and the loss of positive definiteness of the inverse correlation matrix, are investigated under a finite precision arithmetic. First, we prove that the three instability sources do not exist in the BPLS algorithm. Then, the derivations are extended to the combination of the BPLS and the FNTF algorithms and show that no numerical problem will occur due to these instability sources even under a finite precision implementation. Finally, the validity of our investigations is confirmed through computer simulations using a variety of word-length floating-point arithmetic.

2 Combination of BPLS and FNTF Algorithms

The prediction part of the BPLS algorithm for computing the gain vector is given by the following time- and order-update recursive equations:

$$\psi_m(n) = \mathbf{c}_m^T(n-1)\mathbf{u}_m(n) + u(n-m) \quad (1)$$

$$B_m(n) = \lambda B_m(n-1) + \gamma_m(n)\psi_m^2(n) \quad (2)$$

$$\gamma_{m+1}(n) = \frac{\lambda B_m(n-1)}{B_m(n)}\gamma_m(n) \quad (3)$$

$$\mathbf{c}_m(n) = \mathbf{c}_m(n-1) - \gamma_m(n)\psi_m(n)\tilde{\mathbf{k}}_m(n) \quad (4)$$

$$\tilde{\mathbf{k}}_{m+1}(n) = \begin{bmatrix} \tilde{\mathbf{k}}_m(n) \\ 0 \end{bmatrix} + \frac{\psi_m(n)}{\lambda B_m(n-1)} \begin{bmatrix} \mathbf{c}_m(n-1) \\ 1 \end{bmatrix} \quad (5)$$

where $\psi_m(n)$ is the backward a priori prediction error, $B_m(n)$ is the minimum power of $\psi_m(n)$, $\mathbf{c}_m(n)$ is the tap-weight vector of the backward predictor, $\gamma_m(n)$ is the conversion factor, $\tilde{\mathbf{k}}_m(n)$ is the normalized gain vector and $\mathbf{u}_m(n)$ is the input vector.

The initial conditions for the BPLS algorithm are as follows: At time $n = 0$, set $\mathbf{c}_m(0) = \mathbf{0}_m$, $B_m(0) = \delta$, $\tilde{\mathbf{k}}_m(0) = \mathbf{0}_m$ and $\gamma_m(0) = 1$, where $m = 1, 2, \cdots M-1$. At each iteration $n \geq 1$, generate the first-order variables as follows:

$$\tilde{\mathbf{k}}_1(n) = \frac{u(n)}{\lambda \Phi_1(n-1)} \quad (6)$$

$$\gamma_1(n) = \frac{\lambda \Phi_1(n-1)}{\Phi_1(n)} \quad (7)$$

where $\Phi_1(n)$ is the first-order of the correlation matrix that satisfies

$$\Phi_1(n) = \lambda \Phi_1(n-1) + u^2(n) \quad (8)$$

where $\Phi_1(0) = \delta$.

As soon as the gain vector and the predictor of order M are available, we can use the FNTF algorithm to extend the gain vector from order M to N, which can be written as [1]

$$\tilde{\mathbf{k}}_N(n) = \begin{bmatrix} \tilde{\mathbf{k}}_M(n) \\ \mathbf{0}_{N-M} \end{bmatrix} + \sum_{i=0}^{N-M-1} \frac{\psi_M(n-i)}{\lambda B_M(n-i-1)} \begin{bmatrix} \mathbf{0}_i \\ \mathbf{c}_M(n-i-1) \\ 1 \\ \mathbf{0}_{N-M-i-1} \end{bmatrix} \quad (9)$$

$$\frac{1}{\gamma_N(n)} = \frac{1}{\gamma_M(n)} + \sum_{i=0}^{N-M-1} \frac{\psi_M^2(n-i)}{\lambda B_M(n-i-1)} \quad (10)$$

The adaptive filtering is given by the following equations:

$$\alpha(n) = d(n) - \mathbf{w}_N^T(n-1)\mathbf{u}_N(n) \quad (11)$$

$$\mathbf{w}_N(n) = \mathbf{w}_N(n-1) + \gamma_N(n)\tilde{\mathbf{k}}_N(n)\alpha(n) \quad (12)$$

where $\alpha(n)$ is the a priori estimation error, $d(n)$ is the desired signal, $\mathbf{w}_N(n)$ is the tap-weight vector of the adaptive filter.

3 Analysis of Numerical Property

There are mainly three instability sources reported in the structure of the transversal adaptive filters, which include the unstable behavior of the conversion factor, the loss of symmetry and the loss of positive definiteness of the inverse correlation matrix [5],[6]. In this section, numerical analyses of these instability sources in the BPLS and the combination of the BPLS and the FNTF algorithms are presented. Both infinite and finite precision arithmetic are taken into account for the analysis.

3.1 Conversion Factor

3.1.1 BPLS algorithm: The order-update recursion of the conversion factor in the BPLS algorithm is shown by (3). Expanding this equation, we can write

$$\gamma_{m+1}(n) = \prod_{i=1}^{m} \frac{\lambda B_i(n-1)}{B_i(n)}\gamma_1(n) \quad (13)$$

In infinite precision arithmetic, from (2) and (7), $0 \leq \frac{\lambda B_i(n-1)}{B_i(n)} \leq 1$ and $0 \leq \gamma_1(n) \leq 1$. So the following relation can be obtained.

$$0 \leq \gamma_{m+1}(n) \leq \gamma_m(n) \leq \cdots \leq \gamma_1(n) \leq 1 \quad (14)$$

From the derivation shown in Appendix, (14) is also valid in finite precision implementation, Consequently, the conversion factor in the BPLS algorithm will never be negative or exceed unity.

3.1.2 Combination of BPLS and FNTF algorithm: The extension of the conversion factor from order M to N is shown by (10). Notice that the first term on the right side of (10) $1/\gamma_M(n) \geq 1$ and the second term is always greater or equal to zero. So we have $0 \leq \gamma_N(n) \leq 1$.

3.2 Symmetric Property

3.2.1 BPLS algorithm: $\mathbf{P}_m(n)$ in the BPLS algorithm is inherently symmetric. This can be shown by

$$\tilde{\mathbf{k}}_m(n) = \frac{1}{\lambda}\mathbf{P}_m(n-1)\mathbf{u}_m(n) = (\tilde{\mathbf{k}}_m^T(n))^T = \frac{1}{\lambda}(\mathbf{u}_m^T(n)\mathbf{P}_m(n-1))^T \quad (15)$$

Apparently, (15) is also true in finite precision implementation.

3.2.2 Combination of BPLS and FNTF algorithm: Rewrite the summations on the right side of (9) as

$$\frac{1}{\lambda}\mathbf{P}'_N(n-1)\mathbf{u}_N(n)$$
$$= \sum_{i=0}^{N-M-1} \frac{1}{\lambda B_M(n-i-1)} \begin{bmatrix} 0_i \\ \mathbf{c}_M(n-i-1) \\ 1 \\ 0_{N-M-i-1} \end{bmatrix}$$
$$\cdot[0_i \;\; \mathbf{c}_M(n-i-1) \;\; 1 \;\; 0_{N-M-i-1}] \cdot \mathbf{u}_N(n) \quad (16)$$

Apparently, $\mathbf{P}'_N(n-1) = \left(\mathbf{P}'_N(n-1)\right)^T$ is satisfied even under a finite-precision implementation. So the symmetric property of the inverse correlation matrix of order N is remained.

3.3 Positive Definiteness

The positive definiteness of $\mathbf{P}_m(n-1)$ can be defined as

$$\mathbf{u}_m^T(n)\mathbf{P}_m(n-1)\mathbf{u}_m(n) > 0 \qquad (17)$$

where $\mathbf{u}_m(n) \neq 0$ is the input vector.

3.3.1 BPLS algorithm: Left multiplying (5) by $\mathbf{u}_{m+1}^T(n)$ and recognizing that $\tilde{\mathbf{k}}_{m+1}(n) = \frac{1}{\lambda}\mathbf{P}_{m+1}(n-1)\mathbf{u}_{m+1}(n)$, we get

$$\frac{1}{\lambda}\mathbf{u}_{m+1}^T(n)\mathbf{P}_{m+1}(n-1)\mathbf{u}_{m+1}(n)$$
$$= \frac{1}{\lambda}\mathbf{u}_m^T(n)\mathbf{P}_m(n-1)\mathbf{u}_m(n) + \frac{\psi_m(n)}{\lambda B_m(n-1)}$$
$$\cdot(\mathbf{u}_m^T(n)\mathbf{c}_m(n-1) + u(n-m)) \qquad (18)$$

Using (1), (18) can be rewritten as

$$\mathbf{u}_{m+1}^T(n)\mathbf{P}_{m+1}(n-1)\mathbf{u}_{m+1}(n)$$
$$= \mathbf{u}_m^T(n)\mathbf{P}_m(n-1)\mathbf{u}_m(n) + \frac{\psi_m^2(n)}{B_m(n-1)} \quad (19)$$

Since $\frac{\psi_m^2(n)}{B_m(n-1)}$ is always positive, we can conclude that if $\mathbf{P}_m(n-1)$ preserves its positive definiteness, then the order-update of $\mathbf{P}_{m+1}(n-1)$ by using (5) remains positive definite.

To analyze the positive definiteness of $\mathbf{P}_{m+1}(n-1)$ under finite-precision implementation, we expand (19) as

$$\mathbf{u}_{m+1}^T(n)\mathbf{P}_{m+1}(n-1)\mathbf{u}_{m+1}(n)$$
$$= u(n)P_1(n-1)u(n) + \sum_{i=1}^{m} \frac{\psi_i^2(n)}{B_i(n-1)} \quad (20)$$

In Appendix, we have proved that the terms on the right side of (20) are always greater or equal to zero. So the nonnegative definiteness of $\mathbf{P}_{m+1}(n-1)$ is held despite finite-precision implementation.

3.3.2 Combination of BPLS and FNTF algorithm: Left multiplying (9) by $\mathbf{u}_N(n)$, we have

$$\mathbf{u}_N^T(n)\mathbf{P}_N(n-1)\mathbf{u}_N(n) = \mathbf{u}_M(n)\mathbf{P}_M(n-1)\mathbf{u}_M(n)$$
$$+ \sum_{i=0}^{N-M-1} \frac{\psi_M^2(n-i)}{B_M(n-i-1)} \qquad (21)$$

Following the same procedure as (20), we can prove that the nonnegative definiteness of $\mathbf{P}_N(n-1)$ is also guaranteed.

The computational load of the combination of the BPLS and the FNTF algorithms is about $\frac{3}{2}M^2+5M+2N$. Since $M \ll N$ is usually satisfied in some applications such as acoustic echo canceler, the computation reduction can be significant.

4 Simulation Results

Computer simulations are done to confirm the validity of our study presented in Sec.3. An adaptive system identification problem is employed. A floating-point arithmetic that consists of an 8-bit exponent and a variable mantissa (including a sign bit) is used for the simulation. A speech signal as shown in Fig.1(a) is used as the input. The unknown system is supposed to be a 10-th order butterworth IIR filter. The number of tap weights used in the adaptive filter is 50. The initial parameter $\delta = 0.1$ and the forgetting factor $\lambda = 0.95$ are used.

The simulation results of three main instability sources effects are shown in Fig.1(b)-(g). From these results, we make the following observations:

- The conversion factors in both the BPLS and BPLS+FNTF algorithms are always in the range between 0 and 1 even though a low bit mantissa is used.
- The symmetric property of $\mathbf{P}_N(n-1)$ is remained in both the BPLS and the BPLS+FNTF algorithms.
- No loss of positive definiteness in the BPLS or the BPLS+FNTF algorithms occurs under a finite precision implementation.

These observations have confirmed the validity of our analysis presented in Sec.3.

Without the effects of three main instability sources, the numerical performance of the BPLS+FNTF algorithms is expected to be much improved. This is virtually true through computer simulations. Figure 2 shows the residual error of some different combinations computed by using a

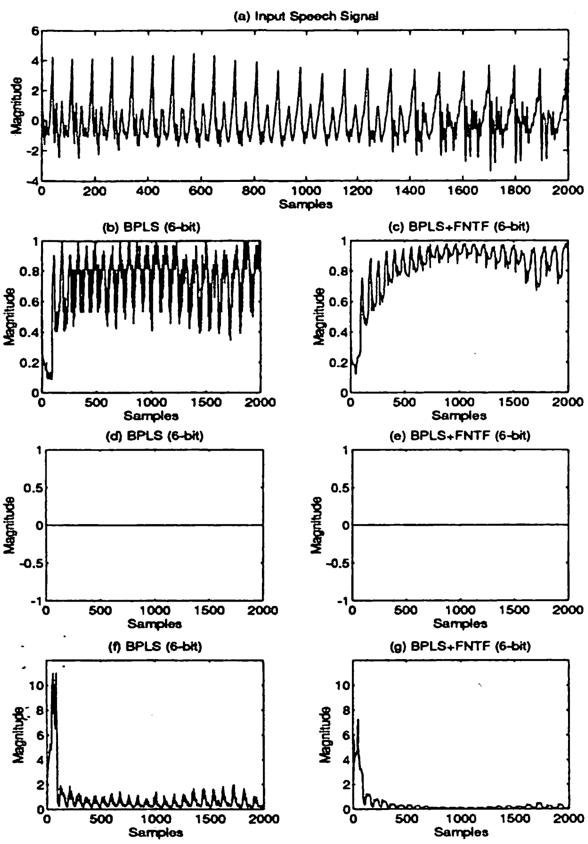

Fig. 1. Simulation conditions: $N = 50$ used for BPLS and $N = 50, M = 10$ used for BPLS+FNTF, 6-bit mantissa. (a) Input speech signal, (b),(c) Conversion factor $\gamma_N^q(n)$ of BPLS and BPLS+FNTF, (d),(e) Symmetric property of BPLS and BPLS+FNTF computed by using $||\mathbf{P}_N^q(n-1)\mathbf{u}_N(n) - \left(\mathbf{u}_N^T(n)\mathbf{P}_N^q(n-1)\right)^T||$. (f),(g) Positive definiteness of BPLS and BPLS+FNTF computed by using $\mathbf{u}_N^T(n)\mathbf{P}_N^q(n-1)\mathbf{u}_N(n)$.

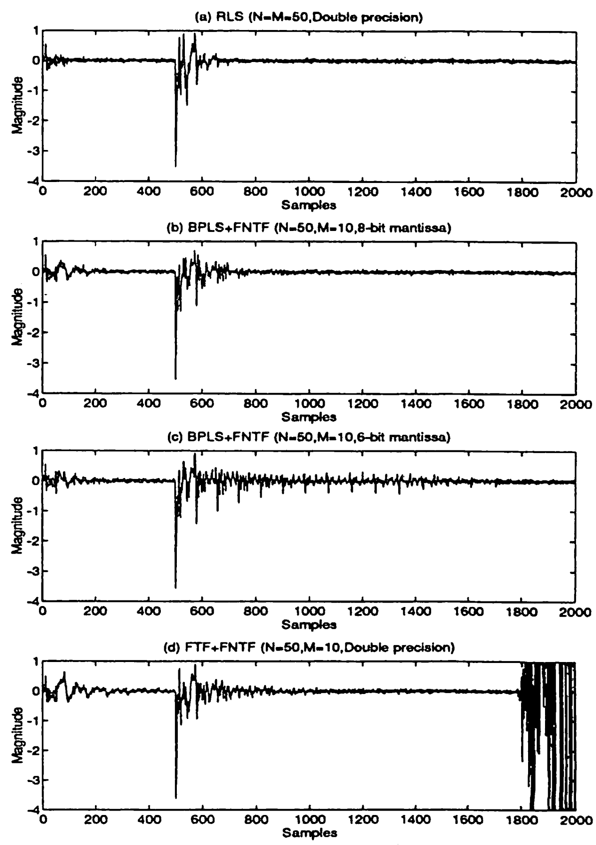

Fig. 2. Simulation conditions: Unknown system $N = 50$ and changes at 500 samples. (a) RLS, $N = M = 50$, double precision, (b) BPLS+FNTF, $N = 50$, $M = 10$, 8-bit mantissa, (c) BPLS+FNTF, $N = 50$, $M = 10$, 6-bit mantissa, (d) FTF+FNTF, $N = 50$, $M = 10$, double precision.

variety of word-length mantissa bits. As expected, the numerical performance of the BPLS + FNTF algorithm is very robust to round-off errors produced by finite-precision implementations. On the other hand, the FTF+FNTF algorithm is unstable even under the double-precision implementation.

5 Conclusion

A numerical study on the combination of the BPLS and the FNTF algorithms has been presented. Finite-precision analysis of three main instability sources, including the overrange of the conversion factor, the loss of symmetry and the loss of positive definiteness of the inverse correlation matrix, has been carried out. The validity of the analysis has been confirmed through computer simulations. It has been shown that the three main instability do not exist in both the BPLS and its combination with the FNTF algorithms. This leads to a much improved numerical performance. The combined algorithm can be applied to various fields, such as acoustic echo canceler, to provide a fast convergence rate and a stable performance with less computation.

References

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Appendix A

In finite-precision (including both floating-point and fixed-point) arithmetic, let $Q[x]$ denote the quantization of x and assume that the dynamic range for computation is large enough so that no overflow error occurs, then from [7], the following conclusions can be obtained:

1. If $0 \leq a \leq 1$ and $b \geq 0$, then $0 \leq Q[ab] \leq b$.
2. If c is a real variable, then $Q[c^2] \geq 0$.
3. If $a \geq b \geq 0$ and $a \neq 0$, then $0 \leq Q[\frac{b}{a}] \leq 1$.
4. If $a \geq 0$ and $b \geq 0$, then $Q[Q[a]+b] \geq Q[a] \geq 0$.

Based on the above conclusions, we prove some results shown in Sec.3. Assume that we have the following recursive equation

$$\alpha(n) = \lambda\alpha(n-1) + \rho(n)\theta^2(n) \qquad (A\cdot 1)$$

with $0 \leq \lambda \leq 1$ and $\alpha(0) = \delta > 0$. The implementation of (A.1) under a finite-precision arithmetic is

$$\alpha^q(n) = Q\Big[Q[\lambda\alpha^q(n-1)] + Q[\rho(n)Q[\theta^2(n)]]\Big] \qquad (A\cdot 2)$$

We want to prove that if $\rho(n) \geq 0$, then

$$\alpha^q(n) \geq Q[\lambda\alpha^q(n-1)] \geq 0. \qquad (A\cdot 3)$$

Using the mathematical induction, we first write the initial state as

$$\alpha^q(1) = Q\big[Q[\lambda\delta] + Q[\rho(1)Q[\theta^2(1)]]\big] \geq Q[\lambda\delta] \geq 0. \qquad (A\cdot 4)$$

Assume

$$\alpha^q(n-1) = Q\big[Q[\lambda\alpha^q(n-2)] + Q[\rho(n-1) \cdot Q[\theta^2(n-1)]]\big] \geq Q[\lambda\alpha^q(n-2)] \geq 0 \qquad (A\cdot 5)$$

then we get

$$\alpha^q(n) = Q\big[Q[\lambda\alpha^q(n-1)] + Q[\rho(n)Q[\theta^2(n)]]\big] \geq Q[\lambda\alpha^q(n-1)] \geq 0 \qquad (A\cdot 6)$$

Now we prove that the conversion factor $\gamma_m(n)$ computed under finite-precision arithmetic satisfies

$$0 \leq \gamma^q_{m+1}(n) \leq \gamma^q_m(n) \leq \cdots \leq \gamma^q_1(n) \leq 1 \qquad (A\cdot 7)$$

First, to prove $0 \leq \gamma^q_1(n) \leq 1$, we write the finite-precision implementation of (8) as

$$\Phi^q_1(n) = Q\big[Q[\lambda\Phi^q_1(n-1)] + Q[u^2(n)]\big] \qquad (A\cdot 8)$$

Apparently, (A.8) is the special case of (A.2) when $\rho(n) = 1$. Hence,

$$\Phi^q_1(n) \geq Q[\lambda\Phi^q_1(n-1)] \geq 0 \qquad (A\cdot 9)$$

or from (7)

$$0 \leq \gamma^q_1(n) = Q\Big[\frac{Q[\lambda\Phi^q_1(n-1)]}{\Phi^q_1(n)}\Big] \leq 1 \qquad (A\cdot 10)$$

Then, we use (2) and write

$$B^q_1(n) = Q\big[Q[\lambda B^q_1(n-1)] + Q[\gamma^q_1(n)Q[\psi^2_1(n)]]\big] \qquad (A\cdot 11)$$

Notice that (A.11) has the same form as (A.2). So the following equation

$$0 \leq \beta^q_1(n) = Q\Big[\frac{Q[\lambda B^q_1(n-1)]}{B^q_1(n)}\Big] \leq 1 \qquad (A\cdot 12)$$

is true, resulting

$$0 \leq \gamma^q_2(n) = Q[\beta^q_1(n)\gamma^q_1(n)] \leq \gamma^q_1(n) \leq 1 \qquad (A\cdot 13)$$

Following the same procedure, we can deduce that $B^q_i(n) \geq Q[B^q_i(n-1)] \geq 0$ and hence the final result of (A.7).

To prove the nonnegative definiteness of $\mathbf{P}_{m+1}(n-1)$ in finite-precision implementation, we write (20) as

$$Q[\mathbf{u}^T_{m+1}(n)\mathbf{P}_{m+1}(n-1)\mathbf{u}_{m+1}(n)] = Q\Big[Q[u(n)P_1(n-1)u(n)] + \sum_{i=1}^{m} Q\Big[\frac{Q[\psi^2_i(n)]}{\lambda B_i(n-1)}\Big]\Big] \qquad (A\cdot 14)$$

For the first term on the right side of (A.14), from (6), we have

$$0 \leq Q[u(n)P_1(n-1)u(n)] = Q\Big[\frac{Q[u^2(n)]}{\lambda\Phi^q_1(n-1)}\Big] \leq 1 \qquad (A\cdot 15)$$

For the rest terms on the right side of (A.14), we have $Q[\frac{Q[\psi^2_i(n)]}{\lambda B^q_i(n-1)}] \geq 0$, Therefore, the nonnegative definiteness of $\mathbf{P}_{m+1}(n-1)$ is proved.