An Improved Fast Fourier Transform Algorithm Using Mixed Frequency and Time Decimations

KENJI NAKAYAMA

Abstract—An improved FFT algorithm combining both decimations in frequency and in time is presented. Stress will be placed on a derivation of general formulas for submatrices and multiplicands. Computational efficiency is briefly discussed.

I. Introduction

Many efforts on improving the classical FFT algorithm [1] have been reported, with respect to combinations of radixes [2] and to some multiplicand modifications [3]. DFT matrix decomposition based on the Chinese remainder theorem and fast convolution algorithms have been combined into computationally efficient algorithms [4], [5], which can save multiplications at the expense of additions and a regular structure. In order to preserve these features of the FFT, a mixed decimation FFT (MDFFT) algorithm combining both decimations in frequency (DIF) and in time (DIT) has been proposed [6], [7]. This correspondence provides general formulas for submatrices, multiplicands, and computational complexity.

II. General Radix Decimation

A. Definitions for Matrices

DFT Coefficient Matrix $F(N_O)$: An $N_O \times N_O$ size matrix whose element at the ith row and kth column is given by

$$f(i, k) = \exp\left(-j2\pi \frac{ik}{N_O}\right), \quad j = \sqrt{-1}. \tag{1}$$

Radix M Decimation Matrix $D_M(N)$: An $N \times N$ size matrix whose element at the ith row and kth column is given by

$$\begin{aligned} d_M(i, k) &= 1, \quad k = \left[\frac{iM}{N}\right](1 - N) + iM \\ &= 0, \quad \text{otherwise}. \end{aligned} \tag{2}$$

DIF and DIT are performed by multiplying $F(N)$ by $D_M(N)$ from the left side and the right side, respectively. $[x]$ means the maximum integer not exceeding x.

Submatrix of MDFFT $G(N_O, N, k_1, l_1, k_2, l_2)$: An $N \times N$ size matrix whose element at the ith row and kth column is given by

$$g(i, k) = \exp\left\{-j2\pi\left(\frac{ik}{N} + \frac{k_1 + il_1}{N_O} + \frac{k_2 + kl_2}{N_O}\right)\right\}. \tag{3}$$

B. General Radix Decimation

A submatrix obtained through the mixed decimation is expressed using the matrix G. First, reformation of G after DIF and DIT are separately stated here.

Theorem 1—Decimation in Frequency: $G(N_O, N, k_1, l_1, k_2, l_2)$ is broken down by multiplying $D_M(N)$ from the left side as follows:

$$D_M(N)\, G(N_O, N, k_1, l_1, k_2, l_2) = \begin{bmatrix} G_{f,0} & & & 0 \\ & G_{f,1} & & \\ & & \ddots & \\ 0 & & & G_{f,M-1} \end{bmatrix} \cdot \begin{bmatrix} W_0 & W_0 & \cdots & W_0 \\ W_0 & W_1 & \cdots & W_{M-1} \\ \vdots & \vdots & & \vdots \\ W_0 & W_{M-1} & \cdots & W_1 \end{bmatrix} \cdot \begin{bmatrix} A_{f,0} & & & 0 \\ & A_{f,1} & & \\ & & \ddots & \\ 0 & & & A_{f,M-1} \end{bmatrix} \tag{4}$$

$$G_{f,s} = G(N_O, N/M, k_1 + sl_1, Ml_1, k_2, l_2 + sN_O/N) \tag{5}$$

$$W_s = \exp(-j2\pi s/M) I(N/M) \tag{6}$$

$$A_{f,s} = \alpha_f^s I(N/M) \tag{7a}$$

$$\alpha_f = \alpha^{l_2} \tag{7b}$$

$$\alpha = \exp\left(-j2\pi \frac{N}{MN_O}\right). \tag{7c}$$

Proof: An element of $D_M(N)G$ at the $(m, sN/M)$th row and kth column becomes

$$\begin{aligned} g_M\left(m + s\frac{N}{M}, k\right) &= \exp\left\{-j2\pi\left(\frac{(s + mM)k}{N} \right.\right. \\ &\quad \left.\left. + \frac{k_1 + (s + mM)l_1}{N_O} + \frac{k_2 + kl_2}{N_O}\right)\right\}. \end{aligned} \tag{8}$$

Thus, elements of G at the $(s + mM)$th row are moved to the $(m + sN/M)$th row. The above equation is further rewritten as

$$\begin{aligned} g_M\left(m + s\frac{N}{M}, k\right) &= \exp\left\{-j2\pi\left(\frac{mk}{(N/M)} + \frac{(k_1 + sl_1) + m(Ml_1)}{N_O} \right.\right. \\ &\quad \left.\left. + \frac{k_2 + k(l_2 + sN_O/N)}{N_O}\right)\right\}. \end{aligned} \tag{9}$$

A matrix having the above element at the mth row and kth column becomes

$$G_{f,s} = G(N_O, N/M, k_1 + sl_1, Ml_1, k_2, l_2 + sN_O/N). \tag{10}$$

Since an element $g_M(m + sN/M, k + rN/M)$ is separated into

$$\begin{aligned} g_M\left(m + s\frac{N}{M}, k + r\frac{N}{M}\right) &= \exp\left(-j2\pi \frac{sr}{M}\right) \\ &\quad \cdot \exp\left(-j2\pi \frac{Nl_2}{MN_O} r\right) g_M\left(m + s\frac{N}{M}, k\right). \end{aligned} \tag{11}$$

A matrix having the above element at the mth row and kth column can be also transformed into the following separated form:

$$G_{f,s,r} = w^{sr} \cdot \alpha_f^r \cdot G_{f,s} \tag{12a}$$

$$w = \exp(-j2\pi/M) \tag{12b}$$

$$\alpha_f = \alpha^{l_2}. \tag{12c}$$

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The author is with C & C Systems Research Laboratories, NEC Corporation, 4-1-1, Miyazaki, Miyamae-Ku, Kawasaki, 213 Japan.

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Theorem 2: $G(N_O, N, k_1, l_1, k_2, l_2)$ is broken down by multiplying $D_M(N)$ from the right side as follows:

$$G(N_O, N, k_1, l_1, k_2, l_2)\, D_M(N) = \begin{bmatrix} A_{t,0} & & 0 \\ & A_{t,1} & \\ 0 & \ddots & A_{t,M-1} \end{bmatrix} \begin{bmatrix} W_0 & W_0 & \cdots & W_0 \\ W_0 & W_1 & \cdots & W_{M-1} \\ \vdots & \vdots & & \vdots \\ W_0 & W_{M-1} & \cdots & W_1 \end{bmatrix} \cdot \begin{bmatrix} G_{t,0} & & 0 \\ & G_{t,1} & \\ 0 & \ddots & G_{t,M-1} \end{bmatrix} \quad (13)$$

$$G_{t,s} = G(N_O, N/M, k_1, l_1 + sN_O/N, k_2 + sl_2, Ml_2) \quad (14)$$

$$A_{t,s} = \alpha_t^s I(N/M) \quad (15a)$$

$$\alpha_t = \alpha^{l_1}. \quad (15b)$$

Proof: Theorem 2 is proved by exchanging row and column in the previous proof.

III. General Formulas for Submatrices and Multiplicands

There exist degrees of freedom for ordering DIF and DIT. One efficient approach is to use both decimations alternately, because of its regular structure. For this reason, the following descriptions are based on the alternate MDFFT.

Theorem 3: Letting $G(N_O, N, k_1(m), l_1(m), k_2(m), l_2(m))$ be $G_{f,s}$, or $G_{t,s}$ at the mth stage, general formulas for its elements are given by

$$l_1(m) = M^{[(m+1)/2]} l_1(0) + \sum_{i=1}^{[m/2]} s_{2i} M^{[(m+1)/2]+i} \quad (16a)$$

$$l_2(m) = M^{[m/2]} l_2(0) + \sum_{i=0}^{[(m-1)/2]} s_{2i+1} M^{[m/2]+i} \quad (16b)$$

$$k_1(2n+2) = k_1(2n+1) = k_1(0) + s_1 l_1(0) + \sum_{j=1}^{n} s_{2j+1} \left\{ M^j l_1(0) + M^{j-1} \sum_{i=1}^{j} s_{2i} M^i \right\}, \quad m = 2n+1, 2n+2 \quad (17a)$$

$$k_2(2n+1) = k_2(2n) = k_2(0) + \sum_{j=1}^{n} s_{2j} \left\{ M^j l_2(0) + M^{j-1} \sum_{i=0}^{j-1} s_{2i+1} M^i \right\}, \quad m = 2n, 2n+1. \quad (17b)$$

It is assumed that decimation starts from DIF. A parameter s_i is identical with s in Theorems 1 and 2, and i indicates the stage number.

Proof:

$l_1(m)$: DIF and DIT are employed at odd and even stages, respectively. From Theorems 1 and 2, the following recurrence formulas are obtained:

$$l_1(2n) = l_1(2n-1) + s_{2n} M^{2n-1} \quad (18a)$$

$$l_1(2n+1) = Ml_1(2n). \quad (18b)$$

They are separated for odd and even stages as follows:

$$l_1(2n+1) = Ml_1(2n-1) + s_{2n} M^{2n} \quad (19a)$$

$$l_1(2n+2) = Ml_1(2n) + s_{2n+2} M^{2n+1}. \quad (19b)$$

Furthermore, these equations converge the following closed forms:

$$l_1(2n+1) = M^{n+1} l_1(0) + \sum_{i=1}^{n} s_{2i} M^{n+i} \quad (20a)$$

$$l_1(2n+2) = M^{n+1} l_1(0) + \sum_{i=1}^{n+1} s_{2i} \cdot M^{n+i}. \quad (20b)$$

These expressions can be combined into a single formula given by (16a).

$l_2(m)$: Recurrence formulas for $l_2(m)$ at odd and even stages are

$$l_2(2n+1) = Ml_2(2n-1) + s_{2n+1} M^{2n} \quad (21a)$$

$$l_2(2n+2) = Ml_2(2n) + s_{2n+1} M^{2n+1}. \quad (21b)$$

Equation (16b) can be derived in a similar way as that for $l_1(m)$.

$k_1(m)$: From Theorems 1 and 2, the recurrence formulas are obtained.

$$k_1(2n+1) = k_1(2n) + s_{2n+1} \cdot l_1(2n) \quad (22a)$$

$$k_1(2n+2) = k_1(2n+1). \quad (22b)$$

Equation (22a) is modified into an odd stage equation as

$$k_1(2n+1) = k_1(2n-1) + s_{2n+1} l_1(2n). \quad (23)$$

A closed form for $k_1(2n+1)$ becomes

$$k_1(2n+1) = k_1(0) + s_1 l_1(0) + \sum_{j=1}^{n} s_{2j+1} \cdot l_1(2j). \quad (24)$$

Based on this, a general formula (17a) can be derived. $k_2(2n+2)$ is directly obtained from $k_1(2n+1)$ as stated in (22b).

$k_2(m)$: Recurrence formulas for $k_2(m)$ become

$$k_2(2n+1) = k_2(2n) \quad (25a)$$

$$k_2(2n+2) = k_2(2n+1) + s_{2n+2} \cdot l_2(2n+1). \quad (25b)$$

A derivation process of (17b) is also the same as that for $k_1(m)$.

Theorem 4: Letting $\alpha_f(m)$ and $\alpha_t(m)$ be α_f and α_t at the mth stage, general formulas for them are given by

$$\alpha_f(m) = \exp\left(-j2\pi l_2(m-1)/M^m\right) \quad (26a)$$

$$\alpha_t(m) = \exp\left(-j2\pi l_1(m-1)/M^m\right). \quad (26b)$$

Proof: This theorem is easily proved from (7) and (15).

A structure for the radix 2 alternate MDFFT is illustrated in Fig. 1.

IV. Computational Complexity

The proposed MDFFT requires multiplications for $\alpha_f^{sm}(m)$ and $\alpha_t^{sm}(m)$, and $G_{f,s_1s_2\cdots s_L}$ or $G_{t,s_1s_2\cdots s_L}$ at the final stage, where $L = \log_2 N_O$. The number of multiplications is obtained by substituting values 0 and 1 or 0, 1, 2, and 3 into s_i in the general formulas for the multiplicands. Coefficients $e^{-j2\pi/n}$ require zero, two, and three real multiplications for $n \leqq 4$, $n = 8$, and $n > 8$, respectively [6]. Additions are required in the complex multiplications and DIF and DIT stages. The number of real multiplications for radix 2 MDFFT becomes

$$O_{\text{MDFFT}} = \tfrac{3}{2} N \log_2 N - 7N + 10\sqrt{N} - 4. \quad (27)$$

Input data are assumed to be complex. On the other hand, the number of multiplications for radix 2 FFT algorithm is given by

$$O_{\text{FFT}} = \tfrac{3}{2} N \log_2 N - 5N + 8 \quad (28)$$

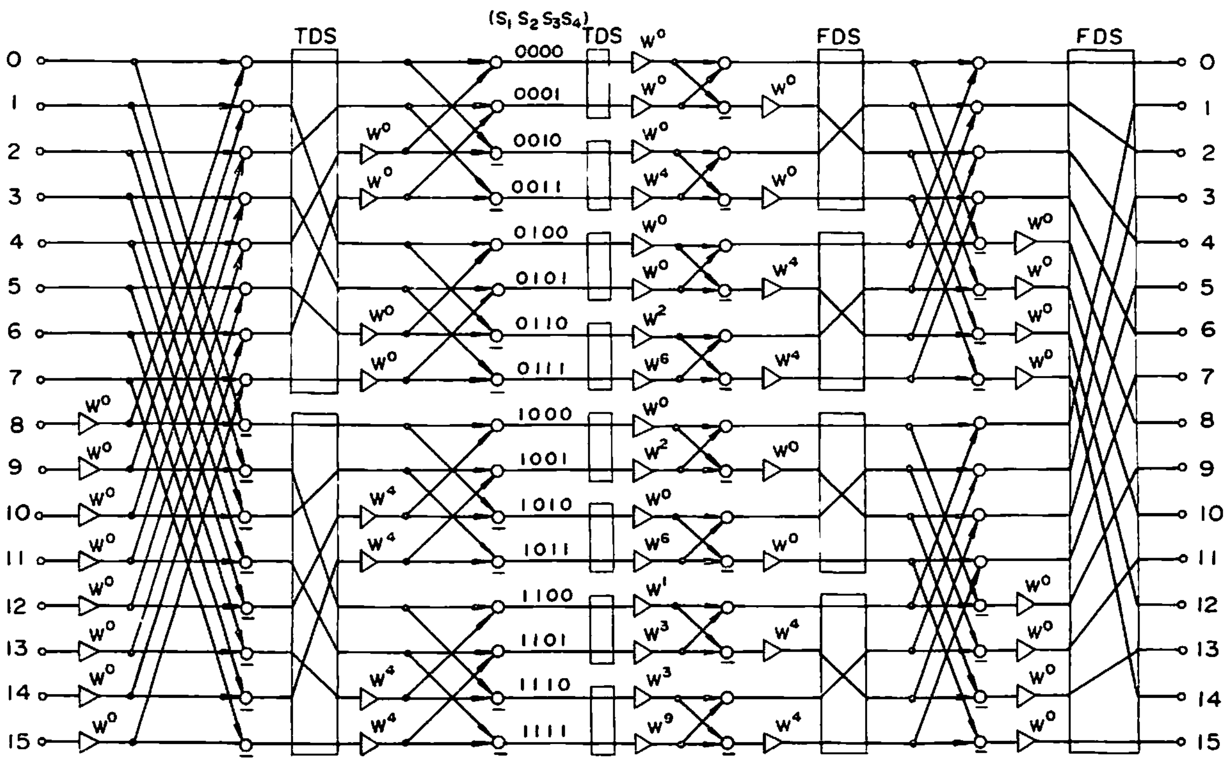

Fig. 1. Block diagram for 16 point radix 2 alternate MDFFT, where $w = \exp(-j2\pi/16)$.

under the same condition. From (27) and (28), MDFFT can save multiplications by 15-20 percent from the FFT algorithm. Furthermore, additions are slightly reduced. These reductions are also obtained for a radix 4 structure.

V. Conclusion

General formulas for submatrices and multiplicands appearing in MDFFT have been presented. Computational efficiency has been briefly discussed.

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